Centre for Theoretical Physics & Natural Philosophy
"Nakhonsawan Studiorum for Advanced Studies" - NAS

End-Semester Examination

Ph. D. Coursework, NAS-MUNA

Symmetries & Lie Algebra in Physics

(NWTP 702)

Instructor: Kumar Abhinav

Date: May 9, 2025

Time: 10:00 - 13:00 hrs | Semester 2/2024 | Total marks: 30

Instructions

I. Attempt **any 3** out of 4 questions.

II. Use **ONLY** your class notebook(s) and no internet.

III. Do NOT use red ink.

IV. Do NOT waste time writing the questions on the answer-sheet.

V. Ask for clarifications if you do not understand any question.

VI. Individual marks are given in parentheses. Answer according to the marks.

Questions:

1. Consider an irreducible representation of $SU(2)$. The only Casimir of this system is given by,

$$J^2 = J_1^2 + J_2^2 + J_3^2, \quad \left[J^2,\, J_i\right] = 0 \,\forall\, i = 1, 2, 3.$$

a) Construct a basis state for this system and explain its eigenvalues. [2]

b) Show that, using the $su(2)$ Lie algebra and the raising/lowering operators,

$$L_\pm = \frac{1}{\sqrt{2}} \left(J_1 \pm i J_2\right),$$

that the eigenvalue of J^2 is $j(j+1)$, j being the highest eigenvalue of J_3. [5]

c) Now consider the following algebra:

$$\left\{a\,,\, a^\dagger\right\} = aa^\dagger + a^\dagger a = 1, \quad \{a\,,\, a\} = 0 = \left\{a^\dagger,\, a^\dagger\right\},$$

corresponding to a Hamiltonian $H = a^\dagger a$. Show that it is a 2-level system. [3]

2. a) Describe the structure of the Lorentz group. [2]

b) Show that,

$$\hat{R}(\omega) = \frac{1}{2}\omega^{\mu\nu} M_{\mu\nu},$$

generates rotations in the Minkowski space. Here $\omega^{\mu\nu}$ are the spin connections whereas $M_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$. [5]

c) Calculate the commutator,

$$\left[M_{\mu\nu}\,,\, M_{\rho\lambda}\right],$$

where $M_{\mu\nu}$ is explained in part *b*) above. [3]

3. a) Sow that the global $U(1)$ transformation,

$$\varphi(x) \to e^{i\alpha}\varphi(x),$$

with α being a constant parameter, is not a symmetry for a massive but real Klein-Gordon field but it is for the complex counterpart. The massive Klein-Gordon field satisfies the equation,

$$\left(\partial_\mu \partial^\mu + m^2\right)\varphi(x) = 0.$$

[3]

b) Show that the local $U(1)$ transformation,

$$\varphi(x) \to e^{i\theta(x)}\varphi(x),$$

can be a symmetry of a complex Klein-Gordon scalar field only if one transitions as,

$$\partial_\mu \to \partial_\mu + iA_\mu(x),$$

to the covariant derivative with the gauge field transforming as,

$$A_\mu(x) \to A_\mu(x) - \partial_\mu\theta(x),$$

under the same gauge transformation. [5]

c) Construct a kinetic term for the gauge field $A_\mu(x)$ in the Lagrangian that is gauge-invariant. [2]

4. a) Consider a Lie algebra,

$$[X_a\,,\,X_b] = if_{abc}X_c.$$

Construct the adjoint representation and show that it satisfies the Lie algebra using the Jacobi identity:

$$f_{ade}f_{bcd} + f_{bde}f_{cad} + f_{cde}f_{abd} = 0.$$

[3]

b) Consider two spin-3/2 particles (they have no internal structures; the net spin can not be resolved into constituent spins). Using the Young tableaux find out how many states in total are there for various representations of this system. Verify this with a numerical relation of the type:

$$2 \times 3 = 5 + 1,$$

and explain. [4+1]

c) Define roots and weights of a Lie algebra. [1+1]

Best wishes